

























## Detection Limits of Low-mass, Long-period Exoplanets Using Gaussian Processes Applied to HARPS-N Solar RVs

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### ABSTRACT

Radial velocity (RV) searches for Earth-mass exoplanets in the habitable zone around Sun-like stars are limited by the effects of stellar variability on the host star. In particular, suppression of convective blueshift and brightness inhomogeneities due to photospheric faculae/plage and starspots are the dominant contribution to the variability of such stellar RVs. Gaussian process (GP) regression is a powerful tool for modeling these quasi-periodic variations. We investigate the limits of this technique using 800 days of RVs from the solar telescope on the HARPS-N spectrograph. These data provide a well-sampled time series of stellar RV variations. Into this data set, we inject Keplerian signals with periods between 100 and 500 days and amplitudes between 0.6 and 2.4 m s<sup>-1</sup>. We use GP regression to fit the resulting RVs and determine the statistical significance of recovered periods and amplitudes. We then generate synthetic RVs with the same covariance properties as the solar data to determine a lower bound on the observational baseline necessary to detect low-mass planets in Venus-like orbits around a Sun-like star. Our simulations show that discovering such planets using current-generation spectrographs using GP regression will require more than 12 years of densely sampled RV observations. Furthermore, even with a *perfect* model of stellar variability, discovering a true exo-Venus with current instruments would take over 15 years. Therefore, next-generation spectrographs and better models of stellar variability are required for detection of such planets.

**Keywords:** exoplanets — techniques: radial velocities, Gaussian processes — Sun: activity — planets and satellites: detection

## 1. INTRODUCTION

State-of-the-art radial velocity (RV) searches for low-mass, long-period exoplanets are limited by signals produced by stellar magnetic variability. An Earth-like planet in orbit around a Sun-like star in its habitable zone induces a reflex RV signal on the order of  $10 \text{ cm s}^{-1}$ . However, the presence of acoustic oscillations, magnetoconvection, large-scale magnetic structures, and other stellar processes induce RV perturbations that can exceed  $1 \text{ m s}^{-1}$  (see Cegla, H. M. (2019); Fischer, D. A. et al. (2016) and references therein). This stellar variability can conceal and even mimic planetary signals in RV surveys, and has resulted in many false detections (e.g., the disproval of CoRoT-7d, Haywood, R. D. et al. (2014); GJ 581d and g, Robertson, P. et al. (2014); Alpha Centauri Bb, Rajpaul, V. et al. (2016)). Furthermore, these stellar processes act on timescales between minutes and months (Kjeldsen, H. & Bedding, T. R. 1995; Giles, H. et al. 2017). For Sun-like stars, the dominant contributions to these intrinsically driven RV perturbations are from the suppression of convective blueshift and brightness inhomogeneities modulated at the rotation period (Meunier, N. et al. 2010; Dumusque, X. et al. 2014). The wide range of timescales and non-trivial correlations between these processes require a sophisticated statistical framework to decouple stellar activity processes from planetary signals. Hall, R. D. et al. (2018) and Haywood, R. D. et al. (2020) study these effects by recovering injected planets of known properties into real data.

In this work, we use state-of-the-art Gaussian process (GP) regression to account for the temporal correlations of rotationally-modulated stellar activity (Dumusque, X. et al. 2017; Damasso, M. et al. 2019). This GP regression is trained on solar data, as measured by a purpose-built solar telescope feeding the HARPS-N spectrograph (Cosentino, R. et al. 2014) operating at the Telescopio Nazionale Galileo (TNG) in the Canary Islands (Dumusque, X. et al. 2015; Phillips, D. F. et al. 2016). In section 2, we first present the solar data and the GP regression along with the resulting fit on the daily-averaged solar RVs. In section 3, we introduce synthetic planets of varying semi-amplitude and orbital period in order to determine the sensitivity of the GP regression to temperate, low-mass planet searches. We conclude in section 4 with an analysis of the baseline of RV observations and model assumptions that are necessary for a true exo-Earth detection.

## 2. METHODS

### 2.1. Data

We take 5 minute disk averaged exposures of the Sun using the solar telescope and the HARPS-N spectrograph at the TNG, and use a baseline of around 800 days of near-continuous, daytime, solar spectra (Dumusque, X. et al. 2015; Phillips, D. F. et al. 2016). For each exposure, the HARPS-N Data Reduction Software (DRS) (Baranne, A. et al. 1996; Sosnowska, D. et al. 2012) computes the barycenter-corrected RV with  $40 \text{ cm s}^{-1}$  single exposure precision and the Mt. Wilson S index (Wilson, O. C. 1968), a measure of stellar magnetic activity derived from chromospheric re-emission in the core of the singly-ionized Ca H and K line cores (Linksy, J. L. & Avrett, E. H. 1970). The resulting RVs are further reduced, as described in Collier Cameron, A et al. (2019), to remove the RV signatures of the solar system planets, effects of differential extinction across the solar disk in the Earth’s atmosphere, and other systematic effects due to the Earth’s orbit around the Sun. To mimic the sampling of a typical stellar observing schedule while preserving the exquisite signal-to-noise ratio (SNR) of the solar telescope (Phillips, D. F. et al. 2016), we compute daily averaged values of each quantity including the RVs, the S index, and the corresponding mean Julian date. While realistic stellar observing schedules would not allow for this level of averaging, which integrates over variability on the minutes and hours timescale, we wish to assess the best-case scenario using an ultra-high SNR dataset.

### 2.2. Gaussian Process Kernel

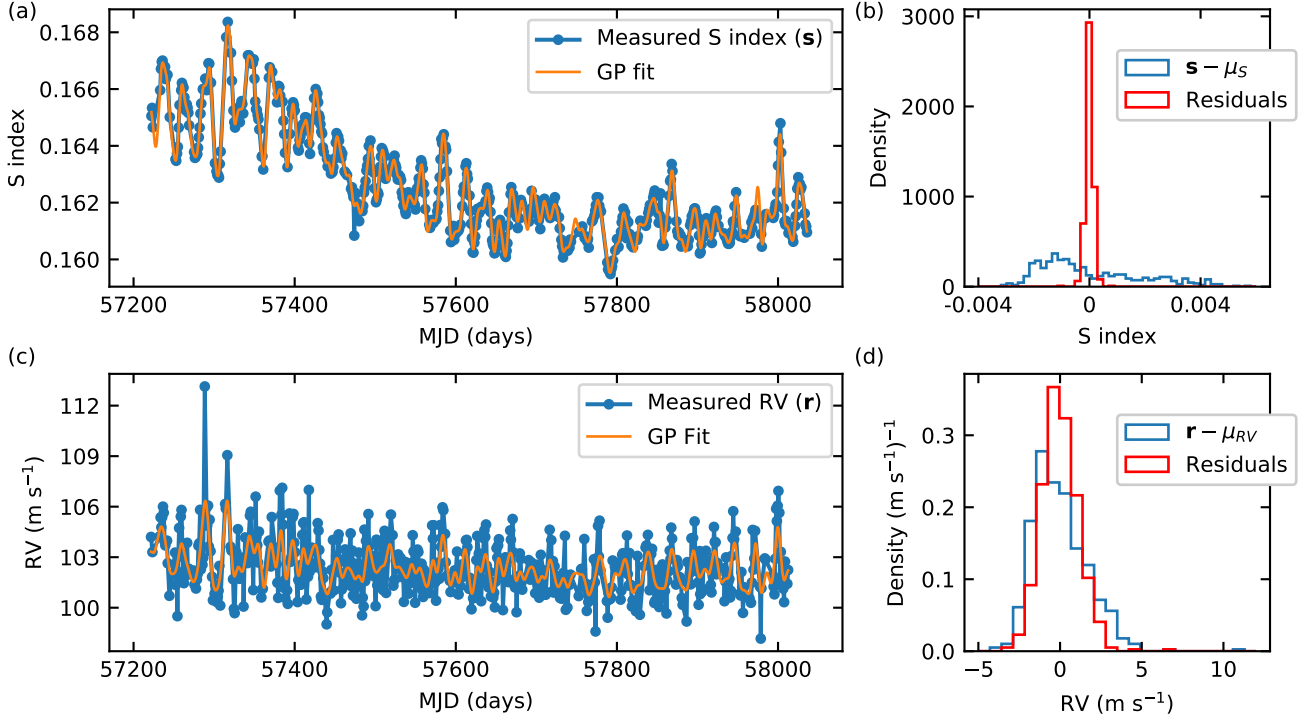
Magnetic variability can be the dominant source of variance for stellar RVs of nearby, bright Sun-like and low-mass stars, reaching levels surpassing  $1 \text{ m s}^{-1}$  (Isaacson, H. & Fischer, D. 2010; Motalebi, F. et al. 2015). Intrinsic stellar variability introduces correlations into the RV time series that are difficult to model deterministically. GP regressions have emerged as a powerful statistical technique that relaxes the assumption of uncorrelated noise by adding nonzero terms to the off diagonal of the data covariance matrix. Usually a kernel function is chosen to describe these covariances as a function of measurement separation time (Rasmussen, C. E. & Williams, C. K. I. 2006; Haywood, R. D. et al. 2014; Rajpaul, V. et al. 2015; Faria, J. P. et al. 2016; Damasso, M. et al. 2019).

Since magnetic activity is well described by the S index and is modulated at the rotation period of the star, we fit the S index to a GP with a quasi periodic (QP) covariance kernel function,  $k_{QP}$ . This kernel function is chosen heuristically to model the known properties of the magnetic activity, yielding the correlation between two measurements at times  $t_i$  and  $t_j$

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**Figure 1.** GP regression fits to  $\sim 800$  days of solar S index and RV data from the solar telescope and HARPS-N. (a) S index (blue) and GP regression fit (orange). (b) Histogram of mean-subtracted S index values (blue) and histogram of the residuals of the GP regression fit (red). (c) Daily average RVs (blue) and GP regression fit (orange). The mean value represents the HARPS-N instrumental offset and carries no physical meaning. (d) Histogram of mean-subtracted RVs (blue) and histogram of the residuals of the GP regression fit (red). The RMS variation is reduced from  $1.65 \text{ m s}^{-1}$  to  $1.14 \text{ m s}^{-1}$ . The relative size of the white noise compared to the correlated noise is higher in the RVs than in the S index, emphasizing the need for a more sophisticated model to account for RV variation.

given by

$$k_{\text{QP}}(t_i, t_j) = \exp\left(-\frac{(t_i - t_j)^2}{2\tau^2} - \frac{1}{2\eta^2} \sin^2\left(\frac{\pi(t_i - t_j)}{P_{\text{rot}}}\right)\right) \quad (1)$$

where  $\tau$  is related to the average lifetime of active regions on the Sun,  $\eta$  is related to the average distribution of activity in the photosphere of the Sun, and  $P_{\text{rot}}$  is approximately the synodic rotation period of the Sun. Grunblatt, S. K. et al. (2015) show this kernel function performs the best among three common GP kernel functions. The GP regression also includes a white noise (WN) term to account for additional uncorrelated noise sources such as instrumental effects. Amplitudes of both the correlated and uncorrelated terms are allowed to vary, leading to a covariance matrix,  $\mathbf{K}_S$ , with components

$$K_{S_{ij}} = \sigma_{\text{QPS}}^2 k_{\text{QP}}(t_i, t_j) + \sigma_{\text{WN,S}}^2 I_{ij} \quad (2)$$

where  $\sigma_{\text{QPS}}$  and  $\sigma_{\text{WN,S}}$  are the two amplitudes, and  $\mathbf{I}$  is the identity matrix.

### 2.3. Fitting the S Index

With a single, time-independent parameter representing the mean value of the S index,  $\mu_S$ , the set of fit parameters

contains six variables:

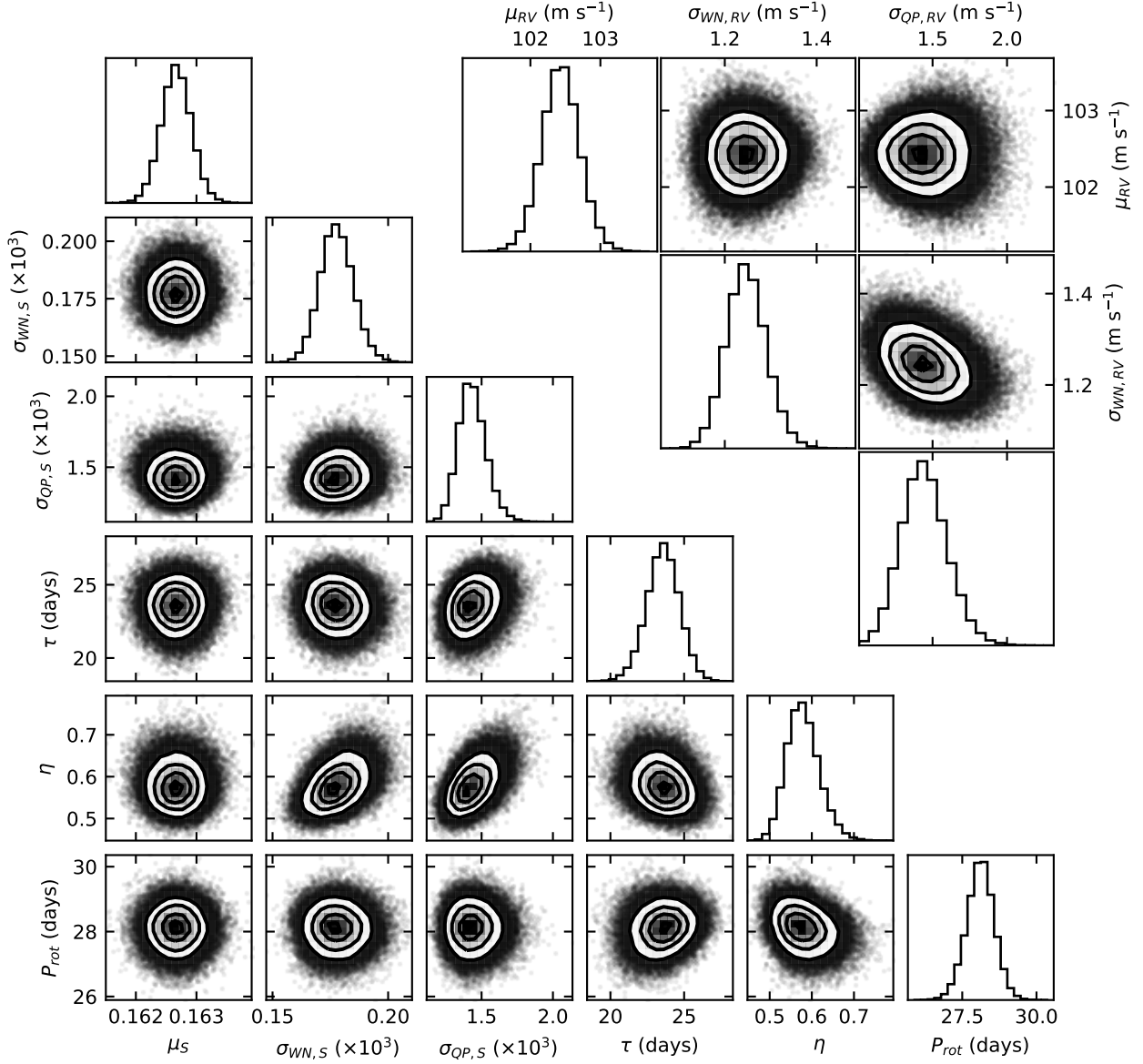
$$\theta_S = \{\mu_S, \sigma_{\text{QPS}}, \sigma_{\text{WN,S}}, \tau, \eta, P_{\text{rot}}\} \quad (3)$$

and the GP likelihood function (Rasmussen, C. E. & Williams, C. K. I. 2006) is given by

$$\mathcal{L}(\theta_S | \mathbf{s}, \mathbf{t}) = \frac{1}{\sqrt{\det(2\pi\mathbf{K}_S)}} \exp\left(-\frac{1}{2}\Delta\mathbf{s}^T \mathbf{K}_S^{-1} \Delta\mathbf{s}\right) \quad (4)$$

where  $\Delta\mathbf{s} = \mathbf{s} - \mu_S$  and  $\mathbf{s}$  is a vector containing the daily averaged S index at times  $\mathbf{t}$  for all the S index values shown in Fig. 1a. Each parameter is assigned a uniform prior probability with upper and lower bounds encompassing realistic values, shown in Table 1. The posterior probability distribution,  $p_S(\theta_S | \mathbf{s}, \mathbf{t})$ , is then proportional to  $\mathcal{L}(\theta_S | \mathbf{s}, \mathbf{t})$  within the prior bounds and 0 otherwise.

We estimate the posterior distribution using an affine-invariant Markov chain Monte Carlo (MCMC) method, implemented with the Python (van Rossum, G. 1995) packages NumPy (Oliphant, T. 2006), SciPy (Jones, E. et al. 2001), emcee (Foreman-Mackey, D. et al. 2013), and george (Ambikasaran, S. et al. 2014). Following Foreman-Mackey, D. et al. (2013) 32 walkers are used to sample the parameter



**Figure 2.** Marginalized distributions of the posterior distribution given by the MCMC samples. The lower left triangle shows samples (and contour lines) from the S index fit and the upper right triangle shows samples (and contour lines) from the RV fit. Each parameter is unimodal and only slight correlations exist between parameters. Thus both posterior distributions are well behaved.

space, and are initialized with a normal distribution around mean values from a maximum likelihood fit and known properties of the Sun. These values are summarized in Table 1. The first 100 samples are discarded allowing the walkers to converge to the posterior distribution before evaluating an additional 50,000 samples per walker. To remove correlations between samples, they are thinned by keeping one out of every  $n_S$  samples for each walker, where  $n_S = 21$  is the average correlation length of the walkers as estimated by the autocorrelation. This yields a total of 76,032 uncorrelated samples for each parameter.

The resulting parameter estimates,  $\hat{\theta}_S$ , are displayed in the last column of Table 1 as the median value of these uncorrelated samples. The error bars are reported as the 16% and 84% quantiles. The values obtained are consistent with known solar properties. Of note is the active region lifetime,  $\tau$ , which is less than one rotation period despite lifetimes of photospheric faculae typically being greater than six rotation periods. Variability of the distribution of faculae on the solar surface is expected to drive this value down below the lifespan of an individual facular region. The GP regression fit to the S index and a histogram of the residuals are shown in Fig. 1a,b. The variances and covariances of the MCMC sam-

	Fit Parameter	Units	Prior	MCMC Initial Guess	Fit Value
$\theta_S$ (S index fit parameters)	$\mu_S$		$U(0.15, 0.17)$	mean(s) = 0.1626	$0.1627^{+0.0002}_{-0.0002}$
	$\sigma_{QP,S}$		$U(10^{-5}, 10^{-1})$	std(s) = $1.83 \times 10^{-3}$	$1.43^{+0.10}_{-0.09} \times 10^{-3}$
	$\sigma_{WN,S}$		$U(10^{-5}, 10^{-1})$	std(s) = $1.83 \times 10^{-3}$	$0.178^{+0.007}_{-0.007} \times 10^{-3}$
	$\tau$	days	$U(5, 100)$	22	$23.6^{+1.1}_{-1.1}$
	$\eta$		$U(0.1, 0.9)$	0.5	$0.58^{+0.04}_{-0.04}$
	$P_{\text{rot}}$	days	$U(24, 32)$	27	$28.1^{+0.4}_{-0.5}$
$\theta_{RV}$ (RV fit parameters)	$\mu_{RV}$	$\text{m s}^{-1}$	$U(97, 113)$	mean( $\mathbf{r}$ ) = 102.4	$102.4^{+0.3}_{-0.3}$
	$\sigma_{QP,RV}$	$\text{m s}^{-1}$	$U(0.01, 10)$	std( $\mathbf{r}$ ) = 1.65	$1.44^{+0.16}_{-0.15}$
	$\sigma_{WN,RV}$	$\text{m s}^{-1}$	$U(0.01, 10)$	std( $\mathbf{r}$ ) = 1.65	$1.25^{+0.04}_{-0.04}$

**Table 1.** Summary of parameters used for Markov chain Monte Carlo (MCMC) sampling for the S index and RV fits of solar telescope and HARPS-N data. The upper and lower bounds of the uniform priors are given in addition to the initial guesses used for each parameter. The last column shows the resulting median value of the MCMC samples and their corresponding 16% and 84% quantiles as error bars.

ples are shown graphically in the bottom left half of Fig. 2. The largest correlations are the  $(\eta, \sigma_{QP,S})$  and  $(\eta, \sigma_{WN,S})$  pairs with statistically significant Pearson correlation coefficients (Student 1908) of 0.49 and 0.42 respectively. This could indicate an unaccounted for spatial dependence in the magnetic activity. Furthermore, significant correlations exist in the  $(\eta, \tau)$  and  $(\eta, P_{\text{rot}})$  pairs, potentially reflecting the migration of active regions from higher latitudes (and thus longer rotation period) to lower latitudes (and thus shorter rotation period) over the 2.2 years of data. Additionally,  $\sigma_{QP,S}$  increases slightly with  $\tau$ , lending further evidence to a time dependence in the fit parameters as  $\tau$  will be longer for the on average larger, longer-lived active regions earlier in the solar magnetic cycle. These correlations suggest a more sophisticated model may more closely represent the physical mechanisms in the photosphere of the Sun and thus capture more of the RV variation due to activity. Further investigation of these correlations are left to future study.

#### 2.4. Fitting the RVs

We assume that the magnetic activity driving the S index also affects the observed RVs with equivalent correlated variability and thus may be described with equal GP regression parameters:  $\tau$ ,  $\eta$ , and  $P_{\text{rot}}$ . Unlike Rajpaul, V. et al. (2015) who fit the S index and RVs simultaneously, we model the S index and RVs consecutively. We fit the RVs using the same quasiperiodic kernel function of equation (1) with the spot lifetime,  $\tau$ , spot distribution,  $\eta$ , and the rotation period  $P_{\text{rot}}$  fixed at the median value from the S index fit. Because the effects of solar system planets have already been removed from the RVs, we are left with only three fit parameters:

$$\theta_{RV} = \{\mu_{RV}, \sigma_{QP,RV}, \sigma_{WN,RV}\} \quad (5)$$

and the likelihood function becomes

$$\mathcal{L}(\theta_{RV} | \mathbf{r}, \mathbf{t}) = \frac{1}{\sqrt{\det(2\pi\mathbf{K}_{RV})}} \exp\left(-\frac{1}{2}\Delta\mathbf{r}^T\mathbf{K}_{RV}^{-1}\Delta\mathbf{r}\right) \quad (6)$$

where  $\Delta\mathbf{r} = \mathbf{r} - \mu_{RV}$ ,  $\mathbf{r}$  is a vector of the RVs at times  $\mathbf{t}$  and  $\mathbf{K}_{RV}$  is populated with  $\theta_{RV}$  by substituting all ‘S’ subscripts with ‘R’ in equation (2). Priors are shown in Table 1.

The posterior probability,  $p_{RV}(\theta_{RV} | \mathbf{r}, \mathbf{t})$ , is proportional to the likelihood function within the bounds of the prior and 0 otherwise. It is sampled using the same MCMC protocol as with the S index and we observe a correlation length of  $n_{RV} = 11$ . Retaining one of every  $n_{RV}$  samples leads to a total of 145,312 uncorrelated samples. The parameter estimates,  $\hat{\theta}_{RV}$ , are summarized in Table 1 and the resulting GP regression fit and residuals are shown in Fig. 1. The fit reduces the RMS scatter in the data from  $1.65 \text{ m s}^{-1}$  to  $1.14 \text{ m s}^{-1}$ , consistent with Milbourne, T. W. et al. (2019), Miklos, M. et al. (2020), and other solar analyses (see section 5). The marginalized distributions are shown in Fig. 2 and again display a well behaved posterior distribution. The only significant correlation exists between  $\sigma_{QP,RV}$  and  $\sigma_{WN,RV}$  and is negative, which is expected as these parameters will trade off the amount of variation seen in the RVs.

### 3. SENSITIVITY MAP

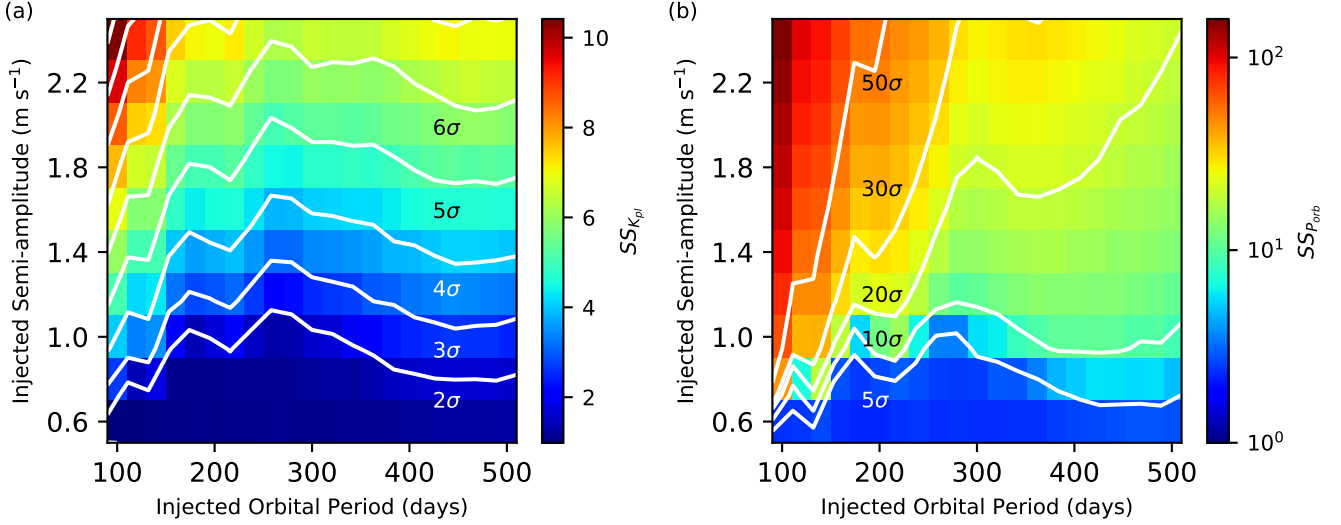
#### 3.1. Synthetic Planet Model

To explore the limits of our GP regression for detecting low-mass, long-period exoplanets, we inject synthetic planets with varying Keplerian parameters into the solar RVs. The general Doppler induced radial velocity,  $v_{RV}(t)$ , of a host star by a companion planet is given by

$$v_{RV}(t) = K_{\text{pl}} [\cos(\omega + \nu(t|t_p, P_{\text{orb}}, e)) + e \cos(\omega)] \quad (7)$$

where  $K_{\text{pl}}$  is the semi-amplitude,  $\omega$  is the argument of periastron,  $e$  is the eccentricity,  $t_p$  is the time of pericenter passage,  $P_{\text{orb}}$  is the orbital period, and  $\nu(t|t_p, P_{\text{orb}}, e)$  is the true anomaly (Perryman, M. 2011). As a best case scenario, we restrict ourselves to circular orbits (i.e.,  $e = 0$ ), which simplifies the Doppler shift to the sine function

$$v_{RV}(t) = K_{\text{pl}} \sin\left(\frac{2\pi t}{P_{\text{orb}}} + \phi\right) \quad (8)$$



**Figure 3.** Sensitivity maps of the recovered (a) semi-amplitude,  $K_{\text{pl}}$ , and (b) orbital period,  $P_{\text{orb}}$ , of synthesized planets using the GP regression with synthetic planetary signals injected into the 800 days of solar RVs. Color bars show the statistical significance ( $SS$ ) of the recovered parameters as defined in equation (11). The white lines in (a) show contours of  $2\sigma, 3\sigma, \dots$  statistical significance and the white lines in (b) show contours of  $5\sigma, 10\sigma, 20\sigma, 30\sigma,$  and  $50\sigma$  statistical significance. Each "pixel" in either image represents one of the 210 simulated planets. The orbital period is generally recovered with a high degree of confidence but the semi-amplitude is only recovered at the  $5\sigma$  level for planets with semi-amplitude above  $1 \text{ m s}^{-1}$ .

where  $\phi$  is an arbitrary phase. A synthetic planet is then generated by choosing values for the semi-amplitude, orbital period, and phase followed by adding  $v_{\text{RV}}(\mathbf{t})$  to the vector of measured solar RVs,  $\mathbf{r}$ , observed at times  $\mathbf{t}$ . The vector of RVs thus undergoes the transformation

$$\mathbf{r} \rightarrow \mathbf{r} + v_{\text{RV}}(\mathbf{t}). \quad (9)$$

The GP regression now contains a mean function given by the addition of the overall mean value of the RVs,  $\mu_{\text{RV}}$ , along with the Keplerian parameters required to describe the injected circular planetary orbit:

$$\theta_{\text{RV}} = \{K_{\text{pl}}, P_{\text{orb}}, \phi, \mu_{\text{RV}}, \sigma_{\text{QP,RV}}, \sigma_{\text{WN,RV}}\} \quad (10)$$

and the likelihood function is still given by (6), with the exception that the vector of fit residuals becomes  $\Delta \mathbf{r} = \mathbf{r} - (\mu_{\text{RV}} + v_{\text{RV}}(\mathbf{t}))$  with  $\mathbf{t}$  the vector of observation times and  $v_{\text{RV}}$  the function defined in Eq. 8.

### 3.2. Retrieval of Injected Signals

Using the techniques of the previous section, we construct a map of detection sensitivities for a range of synthetic, low-mass, long-period planets. Our grid contains 210 injected planets with semi-amplitudes from  $0.6 \text{ m s}^{-1}$  to  $2.4 \text{ m s}^{-1}$  in  $0.2 \text{ m s}^{-1}$  steps and orbital periods from 100 days to 500 days in 20 day steps. The phase of each planet is drawn from a uniform distribution,  $\phi \sim U(-0.1, 0.1)$  radians, and the prior on the phase is uniform from  $-\pi$  to  $\pi$  radians. This is done to avoid numerical instabilities associated with the

phase occurring near the boundary of the prior. The prior on the semi-amplitude allows only positive values less than  $10 \text{ m s}^{-1}$  and the prior on the orbital period is in the range  $(\frac{1}{2}P_{\text{orb}}, 2P_{\text{orb}})$ . For planets detected with a high degree of statistical significance, the priors are uninformative and do not affect the results. However, for the lowest mass planets with  $1\sigma$  detections or less, the priors do constrain the results as described below. These planets set the lower bound on the range of injected semi-amplitudes used in this analysis.

We draw MCMC samples, exploring the semi-amplitude and orbital period linearly, in the same fashion as the previous section with the non-Keplerian parameter priors unchanged. We again define the parameter estimates,  $\hat{\theta}_{\text{RV}}$ , as the median value of the uncorrelated samples and the corresponding lower and upper bounds,  $\hat{\theta}_{\text{R, lower}}$  and  $\hat{\theta}_{\text{R, upper}}$ , as the 16% and 84% quantiles. The statistical significance,  $SS_i$ , is defined as

$$SS_i = \frac{|\hat{\theta}_i|}{\frac{1}{2}(\hat{\theta}_{i, \text{upper}} - \hat{\theta}_{i, \text{lower}})} \quad (11)$$

where  $\theta_i$  is a given parameter from the vector defined in Eq. (10). We plot the statistical significance of the recovered orbital period and semi-amplitude of the 210 synthetic planets in Fig. 3. The orbital period is determined with a high degree of statistical significance, though for semi-amplitudes below  $1 \text{ m s}^{-1}$  this should be taken as an upper bound as the MCMC samples begin to encounter the edges of the uniform prior. The semi-amplitude, however, is much less cer-

tain. For a  $5\sigma$  “discovery” threshold, planets with a semi-amplitude less than  $1 \text{ m s}^{-1}$  would require more observations than the 800 days of solar telescope data used in this analysis. The structure of the contours in Fig. 3(a) is likely due to our non-continuous observing schedule. In particular, Nava, C. et al. (2019) find that activity-driven signals at orbital periods unrelated to either planetary orbital periods or the stellar rotation period can arise from uneven sampling. Thus even the near-daily, long-baseline observing schedule of the solar telescope decreases the semi-amplitude sensitivity (Hall, R. D. et al. 2018). We find agreement within parameter uncertainties between the injected and extracted orbital periods and semi-amplitudes, suggesting no systematic effects induced by the GP regression. In the next section we explore the baseline of observations required to detect sub- $\text{m s}^{-1}$  planets using completely synthetic RVs.

#### 4. SYNTHETIC RVs

To study longer observing baselines than in the previous section, we synthesize not only Keplerian Doppler shifts, but also the solar RVs themselves. We thus extend the observing baseline to determine the requirements for detecting a low-mass, long-period analog. A synthetic planet with the orbital period of Venus (225 days) is injected with varying semi-amplitude. We use the orbital period of Venus to avoid systematic biases associated with measuring any periodic signal at 365 days.

##### 4.1. Observing schedule

The first step to synthesizing solar telescope data is to create a realistic observing schedule to account for bad weather and telescope downtime. We assume a naive model where each calendar month,  $m$ , is assigned an observation probability,  $p_{\text{obs}}(m)$ , which represents the fraction of days that have observations in month  $m$ . These probabilities are estimated using historical solar telescope data with the corresponding fraction of days with 10 or more observations. These probabilities are shown in red in Fig. 4a.

For any given day  $d$  in month  $m$ , a uniform random number,  $r(d) \sim U(0,1)$ , is drawn and compared to  $p_{\text{obs}}(m)$ . If  $r(d) \leq p_{\text{obs}}(m)$ , an observation occurs. This process is repeated for as many consecutive days as needed. Using this process, we create a synthetic 30 year observing schedule mimicking the solar telescope seasonal variations. The resulting synthetic observing schedule is shown in blue in Fig. 4a.

##### 4.2. Observation times

The second step for simulating data is to generate an observation time for each day with an observation. We begin by computing the mean observation time of all exposures in each day from historical solar telescope data, limiting only to

days with at least 10 exposures. This distribution of mean observation times is histogrammed and fit to a Gaussian kernel density estimate (KDE) (Scott, D. W. 2015). Synthetic observation times are drawn from this KDE. Fig. 4b shows the distribution of mean telescope exposure times and the resulting observation times drawn from the KDE. These observation times coupled with the observing days from the previous section completely determine the observing schedule of the simulated data.

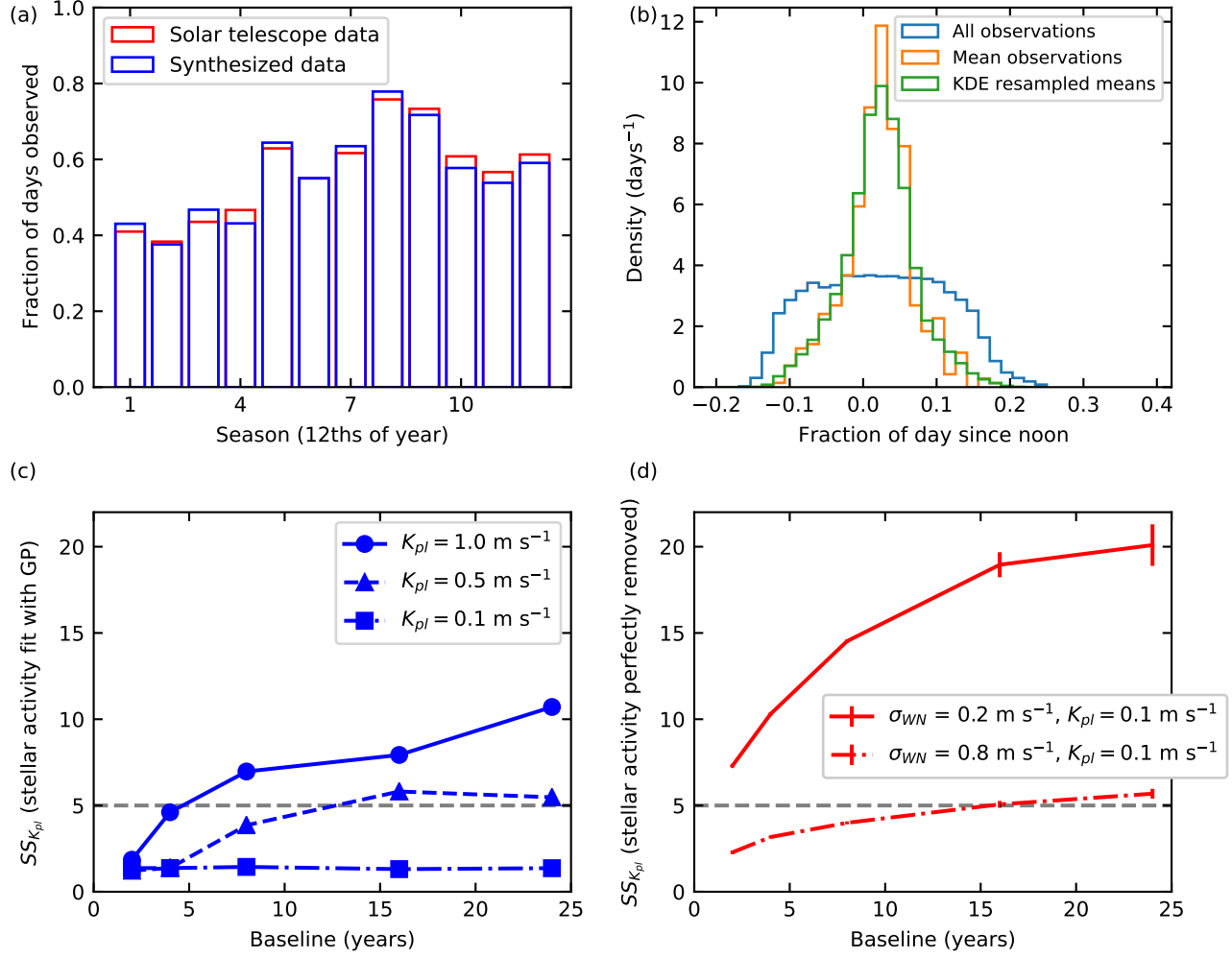
##### 4.3. Synthesized RVs

Finally we synthesize the solar telescope RVs using a GP with the same kernel function given in equation (2). The parameter values used to populate the covariance matrix are taken from the GP regression fit of the solar data as given in Table 1. A random sample of RVs is then drawn from the GP using these parameter values and the synthetic observation times generated in the previous section. Because the solar telescope RVs only cover a fraction of the solar magnetic cycle during solar minimum, we do not model this effect and thus the resulting synthetic RVs represent a best case scenario for this model. Additionally, a Keplerian term given by equation (8) is added with several test semi-amplitudes, an orbital period of 225 days, and a phase drawn randomly in the interval  $(-0.1, 0.1)$  radians.

We then draw MCMC samples and define the parameter estimates and lower and upper bounds as before to determine the statistical significance of the recovered semi-amplitude for baselines of data equal to 2, 4, 8, 16, and 24 years. We repeat this process for semi-amplitudes given by 1, 0.5, and  $0.1 \text{ m s}^{-1}$ . The resulting statistical significances are shown in Fig. 4c. The dashed-dotted curve representing the  $0.1 \text{ m s}^{-1}$  planet represents an upper bound, with the prior probability distribution on the semi-amplitude restricting its value to be positive. Even for a  $0.5 \text{ m s}^{-1}$  planet, we determine that between 10 and 15 years of data are needed to reach the  $5\sigma$  discovery threshold using this GP regression and data similar to that of the HARPS-N solar telescope.

As shown above, a multi-decade temporal baseline is required to detect a temperate, low-mass planet orbiting a Sun-like star using this GP regression. Thus we next assume that we have a direct measure of the magnetic variability of the target star and are able to perfectly remove the effects of variability from the RVs, leaving only the planetary signals and white noise. We can then fit to a sine curve with simple least squares methods. We expect the semi-amplitude uncertainty,  $\sigma_{K_{\text{pl}}}$ , to scale with the measurement sensitivity,  $\sigma_{\text{RV}}$ , divided by the square-root of the number of RV measurements,  $N$  (Cloutier, R. et al. 2018):

$$\sigma_K = \sigma_{\text{RV}} \sqrt{\frac{2}{N}}. \quad (12)$$



**Figure 4.** Generating synthetic RVs based on historical solar telescope observations. (a) Seasonal variation of solar telescope and synthetic observations. Red histogram shows the fraction of days with at least ten 5 minute exposures for 12 evenly spaced sections of a year (does not correspond to calendar months). Blue histogram shows the analogous ratio of seasonal observations from dates synthesized from the red histogram. (b) Histogram of RV observation times given as a fraction of a day since noon UTC: *blue* - all 5 min solar telescope exposures, *orange* - daily averaged observation times for days with at least ten 5 minute exposures, *green* - synthesized observation times using a Gaussian kernel density estimate. (c) Statistical significance (SS), defined in equation (11), of the recovered semi-amplitude,  $K_{pl}$ , using a GP regression for an "exo-Venus" (225 day orbital period) as a function of baseline of data. Solid (circles) and dashed (triangles) curves show the SS for an "exo-Venus" with semi-amplitudes of  $1.0 \text{ m s}^{-1}$  and  $0.5 \text{ m s}^{-1}$  respectively. Dashed-dotted (squares) curve shows an upper bound for a semi-amplitude of  $0.1 \text{ m s}^{-1}$ . Dashed grey line shows the  $5\sigma$  detection threshold. (d) SS for a  $0.1 \text{ m s}^{-1}$  "exo-Venus" with stellar variability perfectly removed (i.e., only white noise). Solid and dashed curves show the SS for white noise amplitudes of  $0.2 \text{ m s}^{-1}$  and  $0.8 \text{ m s}^{-1}$  respectively. Error bars are shown as vertical lines and are too small to be seen for baselines less than 15 years. Again, dashed grey line shows the  $5\sigma$  detection threshold.

To confirm this scaling, we synthesize RV time series comprising simple white noise and a Keplerian Doppler shift of  $K_{pl} = 10 \text{ cm s}^{-1}$  for an injected planet with an orbital period of 225 days (i.e., an exo-Venus). We generate this data for varying baselines and white noise levels of both  $0.8 \text{ m s}^{-1}$  and  $0.2 \text{ m s}^{-1}$ . The statistical significance of the semi-amplitude,  $K_{pl}$ , from least squares fits to these data sets are shown in Fig. 4d and are in good agreement with expectations set by Eq. 12. We emphasize that we need to reduce the white noise to lev-

els approaching  $0.2 \text{ m s}^{-1}$  to reach the  $5\sigma$  detection threshold in only a few years for a true exo-Venus.

## 5. DISCUSSION AND CONCLUSIONS

The GP regression of Sec. 2 reduces the RMS variation of the solar RVs from  $1.65 \text{ m s}^{-1}$  to  $1.14 \text{ m s}^{-1}$ . However, treating stellar variability with this GP regression still requires 10 to 15 years of densely sampled RV observations to detect long-period, low-mass planets. This is much longer than would be expected if the RVs contained purely uncorre-



lated white noise. This result is in line with more physically-motivated techniques. For example, [Milbourne, T. W. et al. \(2019\)](#) used magnetograms and dopplergrams from the Helioseismic and Magnetic Imager (HMI) on board the Solar Dynamics Observatory (SDO) to derive activity-driven RV timeseries. By modeling the HARPS-N RVs using these activity timeseries, they reduced the RV RMS from  $1.65 \text{ m s}^{-1}$  to  $1.21 \text{ m s}^{-1}$ . [Haywood, R. D. et al. \(2020\)](#) reduced the RV RMS to  $0.85 \text{ m s}^{-1}$  by modeling the RVs with a linear combination of the unsigned magnetic flux from HMI and the total solar irradiance, using the  $FF'$  method ([Aigrain, S. et al. 2012](#)). Fitting our GP regression to the unsigned flux may yield a smaller variation in the RV residuals, and this will be investigated in a future work. [Dumusque, X. \(2018\)](#) and [Cretignier, M. et al. \(2020\)](#) used logistic regression and gradient boosting on HARPS-N solar spectra to differentiate activity sensitive and insensitive lines. Computing RVs from these sets of lines reduced the RV RMS to  $0.9 \text{ m s}^{-1}$ . Similarly, [Miklos, M. et al. \(2020\)](#) applied the techniques of [Meunier, N. et al. \(2017\)](#) to the HARPS-N solar RVs to estimate magnetoconvective RV variations. This analysis, however, was unable to reduce the observed RMS RVs.

That the GP regression — a statistical tool — performs similarly to these more physically-motivated analyses demonstrates its power: rather than requiring high-resolution solar images or specially tuned line lists, the GP regression assumes a correlation between the solar S index and observed RVs. However, the fact that the GP analysis and these physically-motivated techniques arrive at the same approximately  $1 \text{ m s}^{-1}$  RV uncertainty level indicates that something else — either another physical process operating on a different timescale or an instrumental systematic — is limiting the performance of these techniques. [Haywood, R. D. et al. \(2020\)](#) discuss some physical effects missing from the current models, to be addressed in future work. However for current data analysis models, the solar dataset used in this work, with nearly 2.5 years of near-daily observations mostly in the decline phase of Cycle 24, represents a best-case scenario; we can therefore only perform  $5\sigma$  recoveries of long-period planets with semi-amplitudes greater than  $1 \text{ m s}^{-1}$ , consistent with the results of the recent community-wide RV challenge of [Dumusque, X. et al. \(2017\)](#).

Our work using synthetic RVs indicates that more observations will not quickly overcome this limit: As shown in Fig. 4, it will take 10-15 years to get  $5\sigma$  on a  $0.5 \text{ m s}^{-1}$  RV signal with a 225 day period (i.e., the orbital period of Venus), and at least 25 years for a  $0.1 \text{ m s}^{-1}$  RV signal. The last panel of this figure also indicates that a *perfect* model of activity-driven correlated variations would not in itself suffice for the rapid detection of an exo-Venus or an exo-Earth: even in the absence of correlated noise, a current-generation spectrograph with a long term stability of about

$0.8 \text{ m s}^{-1}$  would need a 10-15 year observing baseline to reach a  $5\sigma$  detection of an exo-Earth. Successful exo-Earth discovery therefore requires both more sophisticated models of stellar variability and the improved RV precision, long term stability, and dense observational sampling from a next-generation spectrograph ([Wright, J. T. & Robertson, P. 2017](#)) such as ESPRESSO ([Pepe, F. A. et al. 2010](#)), NEID ([Allen, L. E. et al. 2018](#)), EXPRES ([Jurgenson, C. et al. 2016](#)), HARPS3 ([Thompson, S. J. et al. 2016](#)), or G-CLEF ([Szentgyorgyi, A. et al. 2014](#)) on the GMT.

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*Facilities:* TNG: HARPS-N, Solar Telescope

*Software:* Python: Numpy, Scipy, emcee, george

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